A GENERALIZATION OF THE INVARIANT RELATION OF HESS

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Hess' solution [1] of the problem of motion of a body with a stationary point has certain pecularities which have been investigated by Zhukovskii [2], Chaplygin [3] and many others. It was recently discovered that not all the solutions of the problem are stable with respect to generalizations [4]. For example, Kovalevki's solution [5] does not have an analog with the corresponding properties if a gyrostatic moment is applied to the body. Sretenskii [6 and 7] showed, however, that the Hess solution does, in fact, have an analog in the solution of the gyrostat problem. A still more general result was obtained by Kharlamov [8], who pointed out a linear invariant relation in the problem of the motion in a fluid of a body bounded by a multiply connected surface. The solutions of Sretenskii and therefore of Hess are special cases of the Kharlamov solution.

Let us consider a body with a stationary point which has an ellipsoidal cavity completely filled with an ideal fluid in homogeneous vortex motion. This problem is investigated in the monograph by Moiseev and Rumiantsev [9]. The motion of such a body in a gravitational field is described by a system of nine equations. This case also involves a linear invariant relation which is a generalization of the Sretenskii and Hess solutions. In contrast to the latter relations, however, our expression is not a special case of the invariant relation of Kharlamov.

Assuming that the center of gravity of the system under consideration lies on the perpendicular to the circular cross section of the ellipsoid of gyration (the Hess condition), i.e. that

$$z_c = 0, \qquad x_c \sqrt{A(C-B)} - y_c \sqrt{B(A-C)} = 0$$

we can write out Eqs. of motion of the body using the notation of [9],

$$\begin{gathered}
A\frac{d\omega_1}{dt} + A'\frac{d\Omega_1}{dt} + (C-B)\,\omega_2\omega_3 + C'\omega_2\Omega_3 - B'\omega_3\Omega_2 = Mgy_c\gamma_3 \\
B\frac{d\omega_2}{dt} + B'\frac{d\Omega_1}{dt} + (A-C)\,\omega_3\omega_1 + A'\omega_3\Omega_1 - C'\omega_1\Omega_3 = -Mgx_c\gamma_3 \\
C\frac{d\omega_3}{dt} + C'\frac{d\Omega_3}{dt} + (B-A)\,\omega_1\omega_2 + B'\omega_1\Omega_2 - A'\omega_2\Omega_1 = Mg\,(x_c\gamma_2 - y_c\gamma_1) \\
\frac{d\Omega_1}{dt} = 2a^2\left(\frac{\omega_3\Omega_2}{a^2+b^2} - \frac{\omega_2\Omega_3}{c^2+a^2}\right) - 2\Omega_2\Omega_3\frac{a^2(c^2-b^2)}{(a^2+b^2)(a^2+c^2)} \\
\frac{d\Omega_2}{dt} = 2b^2\left(\frac{\omega_1\Omega_3}{b^2+c^2} - \frac{\omega_3\Omega_1}{a^2+b^2}\right) - 2\Omega_3\Omega_1\frac{b^2(a^2-c^2)}{(b^2+c^2)(a^2+b^2)} \\
\frac{d\Omega_3}{dt} = 2c^2\left(\frac{\omega_2\Omega_1}{c^2+a^2} - \frac{\omega_1\Omega_2}{b^2+c^2}\right) - 2\Omega_1\Omega_2\frac{c^2(b^2-a^2)}{(a^2+c^2)(b^2+c^2)} \\
\frac{d\gamma_1}{dt} = \omega_3\gamma_2 - \omega_2\gamma_3, \qquad \frac{d\gamma_2}{dt} = \omega_1\gamma_3 - \omega_3\gamma_1, \qquad \frac{d\gamma_3}{dt} = \omega_2\gamma_1 - \omega_1\gamma_2
\end{gathered} \tag{1}$$

Here A, B, C are the altered moments of inertia of the system; A', B', C' are the differences between the moments of inertia of the fluid and of the equivalent solid ;

a, b, c are the semiaxes of the ellipsoidal cavity; M is the mass of the system.

Let us change variables

$$A\omega_1 = x_1 - AA'c_0 (Cb_0 + Bc_0) N\Omega_1, \qquad B\omega_2 = x_2 - BB'c_0 (Ca_0 + Ac_0) N\Omega_2 \tag{2}$$

$$a_0 = \frac{c^2 - b^2}{c^2 + b^2}, \qquad b_0 = \frac{a^2 - c^2}{a^2 + c^2}, \qquad c_0 = \frac{b^2 - a^2}{b^2 + a^2}$$

$$N^{-1} = ABc_0^2 + C^2 - AC - BC$$

and impose the conditions

$$\begin{vmatrix} A^2 & B^2 & C^2 \\ A & B & C \\ a_0^2 & b_0^2 & c_0^2 \end{vmatrix} = 0$$

$$c_0 (Cb_0 + Bc_0) [B (Ca_0^2 - Ac_0^2) + a_0 b_0 C (C - A)] N - \\ - 0.8\, MC\, [(Cb_0 + Bc_0) - (Ab_0 - Ba_0)] = 0$$

$$c_0 (Ca_0 + Ac_0)[A (Cb_0^2 - Bc_0^2) + a_0 b_0 C (C - B)] N - \\ - 0.8\, MC\, [(Ca_0 + Ac_0) + (Ab_0 - Ba_0)] = 0$$

on the parameters of the system.

Then, converting to the special axes resulting from the rotation of the coordinate system by the angle α

$$\alpha = \operatorname{arctg} \left(\frac{A (C - B)}{B (A - C)}\right)^{1/2}$$

we obtain the first Eq. of system (1) in the form

$$dy_1/dt = my_1 (\omega_3 + n\Omega_3), \qquad y_1 = x_1 \cos\alpha + x_2 \sin\alpha \tag{3}$$

$$n = \frac{B (Ca_0^2 - Ac_0^2) + a_0 b_0 C (C - A)}{(C - A)} N, \qquad m = C \left(\frac{(A - C)(C - B)}{AB}\right)^{1/2}$$

Eq. (3) together with system (1) has a partial integral of the form

$$y_1 = 0$$

or, along the principal axes,

$$Ax_c [\omega_1 + A'c_0 (Cb_0 + Bc_0) N\Omega_1] + By_c [\omega_2 + B'c_0 (Ca_0 + Ac_0) N\Omega_2] = 0$$

This linear invariant relation generalizes the Hess relation and becomes the latter for $a = b = c$.

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A CASE WHERE THE HAMILTON-JACOBI EQUATION IS INTEGRABLE

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Let us consider the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}+\left(\frac{\partial S}{\partial r}\right)^2+F=0 \tag{1}$$

Here S is the action function and F is a given function of the variables r and t. We shall attempt to find the solution of this equation in the form

$$S=S_0+S_1, \quad S_0=X_0(x)T_0(t), \quad S_1=X_1(x)+T_1(t) \tag{2}$$

Here we have introduced the new variable $x=rf(t)$, where $f(t)$ is any doubly differentiable function. Substituting (2) into (1), we obtain

$$\frac{\partial S_0}{\partial t}+\frac{\partial S_0}{\partial x}\frac{x}{f(t)}\frac{df(t)}{dt}+\left(\frac{\partial S_0}{\partial x}\right)^2 f^2(t)+$$
$$+\frac{\partial S_1}{\partial x}\left[\frac{x}{f(t)}\frac{df(t)}{dt}+2\frac{\partial S_0}{\partial x}f^2(t)\right]+\frac{\partial S_1}{\partial t}+\left(\frac{\partial S_1}{\partial x}\right)^2 f^2(t)+F=0 \tag{3}$$

In the latter equation the coefficient of $\partial S_1/\partial x$ is equal to zero provided that

$$S_0=-\frac{x^2}{4f^3(t)}\frac{df(t)}{dt} \tag{4}$$

For this S_0 Eq. (3) yields

$$x^2\left[\frac{1}{2f(t)^6}\left(\frac{df(t)}{dt}\right)^2-\frac{1}{4f(t)^5}\frac{d^2f(t)}{dt^2}\right]+\frac{dT_1(t)}{dt}\frac{1}{f(t)^2}+\left(\frac{dX_1(x)}{dx}\right)^2+\frac{1}{f(t)^2}F=0 \tag{5}$$

The variables in this equation are separable if

$$F=f^2(t)\Psi(x)+f^2(t)\eta(t)+x^2\left[\frac{1}{4f^3(t)}\frac{d^2f(t)}{dt^2}-\frac{1}{2f(t)^4}\left(\frac{df(t)}{dt}\right)^2\right] \tag{6}$$

Here $\Psi(x)$ and $\eta(t)$ are arbitrary functions. If this condition is fulfilled, the total integral Eq. (1) is

$$S=-\frac{x^2}{4f^3(t)}\frac{df(t)}{dt}\pm\int\sqrt{C_1-\Psi(x)}\,dx-\int f^2(t)(C_1+\eta(t))\,dt+C_2$$

where C_1 and C_2 are arbitrary constants.

Specifically, separation of variables in Eq. (5) is possible if

$$\frac{1}{2f^6(t)}\left(\frac{df}{dt}\right)^2-\frac{1}{4f^5(t)}\frac{d^2f}{dt^2}=k, \qquad f=\frac{1}{\sqrt{a(t-b)^2+4k/a}} \qquad F=f^2(t)X(x) \tag{7}$$